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Numerical Solution of the Painlevé VI Equation

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Abstract—A numerical method for solving the Cauchy problem for the sixth Painlevé equation is proposed. The difficulty of this problem, as well as the other Painlevé equations, is that the unknown function can have movable singular points of the pole type; moreover, the equation may have singularities at the points where the solution takes the values 0 or 1 or is equal to the independent variable. The positions of all of these singularities are not a priori known and are determined in the process of solving the equation. The proposed method is based on the transition to auxiliary systems of differential equations in neighborhoods of the indicated points. The equations in these systems and their solutions have no singularities at the corresponding point and its neighborhood. The main results of this paper are the derivation of the auxiliary equations and the formulation of transition criteria. Numerical results illustrating the potentials of this method are presented.

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INTRODUCTION

This publication completes the series of our papers devoted to the numerical solution of the Painlevé equations (see [1–4]). Here, we consider the sixth Painlevé equation (Painlevé VI) in the complex plane with the zero and unity deleted. Its solution is a meromorphic function in the corresponding universal cover, and any point can be a pole of the appropriate solution. Moreover, the equation itself has singularities at the points where the solution $y(x)$ takes the values 0, 1, or x. We say that all of these points are critical. Suppose that, similarly to the other Painlevé equations, the Cauchy problem for Painlevé VI equation must be numerically solved along a given curve. The difficulty of this problem is that the indicated critical points are movable; that is, their positions are not a priori known and depend on the initial data. When solving the Cauchy problem, one should be able to detect a critical point if it is encountered along the given path, determine numerically its location, pass through this point, and find a convenient representation of the solution in its neighborhood.

In this paper, we apply the method that we refer to as the *successive elimination of singularities*. It was successfully applied in [1–4] to solving the Painlevé I–V equations. The method is based on the derivation of auxiliary systems of differential equations that are equivalent to the original equation. The equations in such a system and its solution have no singularities in the corresponding critical point and its neighborhood.

The transition to an auxiliary system of differential equations in a neighborhood of the critical point solves the problem of correctly passing through this point. Moreover, the form of these auxiliary equations allows us to state efficient criteria for transitions to the original equations and in the reverse direction.

The main results of this paper are the derivation of auxiliary equations for all the types of critical points and the formulation of transition criteria for the Painlevé VI equation. We believe that the derived equations, combined with those for the Painlevé I–V equations, can be a useful reference material for persons that would like to solve numerically equations of this type.

Below, we use certain properties of the solutions to the Painlevé equations and more general second-order differential equations with movable singularities. These properties are presented in [5–7].

In what follows, all the variables are assumed to be complex.

1. TYPES OF CRITICAL POINTS OF THE PAINLEVÉ VI EQUATION

The sixth Painlevé equation has the form

$$y'' = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\frac{\beta x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right). \tag{1.1}$$

A solution to this equation can have branch points at $x = 0$ and $x = 1$. It is a meromorphic function in the universal cover of the corresponding punctured plane.

Equation (1.1) depends on the four numerical parameters α, β, γ, and δ. Depending on the values of these parameters, it can have critical points of the following eight types: first- and second-order poles, first- and second-order zeros of $y(x)$, first- and second-order zeros of the function $y(x) - 1$, and first- and second-order zeros of the function $y(x) - x$. We emphasize that the solution is analytic at all points that are not poles.

By way of example, we find out under what relations between the parameters the function $y(x) - x$ can have zeros and what is the order of these zeros.

Let x_* be a zero of $y(x) - x$; that is, $y(x_*) = x_*$. Consider Eq. (1.1) multiplied by $y - x$ at the point x_*. Since $y(x)$ is an analytic function in the neighborhood of x_* (that is, $y(x)$, $y'(x)$, and $y''(x)$ have no singularities at $x = x_*$), the resulting equation implies the relation

$$0 = \frac{1}{2}y'^2(x_*) - y'(x_*) + \delta\frac{y(x_*)(y(x_*)-1)}{x_*(x_*-1)}.$$

Using the equality $y(x_*) = x_*$, we obtain $1/2y'^2 - y' + \delta = 0$. Consequently, if $\delta \neq 1/2$, then $y'(x_*) \neq 1$; hence, x_* is a first-order zero of the function $y(x) - x$. If $\delta = 1/2$, then we have $y'(x_*) = 1$, which means that x_* is a higher order zero of $y(x) - x$. Denote this order by k ($k \geqslant 2$). Let us find the value of k.

Assume that $\delta = 1/2$ and write $y(x)$ in the form

$$y(x) = x + C(x-x_*)^k + O(x-x_*)^{k+1},$$

where $C \neq 0$. Taking into account only the principal terms of the expansions, we obtain

$$y - x = C(x-x_*)^k, \quad y' = 1 + Ck(x-x_*)^{k-1}, \quad y'' = Ck(k-1)(x-x_*)^{k-2}.$$

Now, we substitute these expressions into (1.1) and write the principal terms in the expansions of the individual summands of this equation:

$$\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}\right)y'^2 = \frac{1}{2}\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right);$$

$$\frac{1}{2}\frac{y'^2}{y-x} = \frac{(1+Ck(x-x_*)^{k-1})^2}{2C(x-x_*)^k} = \frac{1}{2C(x-x_*)^k}+\frac{k}{x-x_*}+\frac{Ck^2}{2}(x-x_*)^{k-2};$$

$$-\left(\frac{1}{x}+\frac{1}{x-1}\right)y' = -\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right);$$

$$-\frac{1}{y-x}y' = -\frac{1+Ck(x-x_*)^{k-1}}{C(x-x_*)^k} = -\frac{k}{x-x_*}-\frac{1}{C(x-x_*)^k};$$

$$\frac{\delta y(y-1)}{x(x-1)(y-x)} = \frac{\delta(x+C(x-x_*)^k)(x-1+C(x-x_*)^k)}{Cx(x-1)(x-x_*)^k} = \frac{1}{2}\frac{1}{C(x-x_*)^k}+\frac{1}{2}\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right).$$

The other summands are higher order terms.

By adding these expressions, we arrive at the relation

$$Ck(k-1)(x-x_*)^{k-2} = \frac{1}{2}\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right)+\frac{1}{2C(x-x_*)^k}+\frac{k}{x-x_*}$$
$$+\frac{Ck^2}{2}(x-x_*)^{k-2}-\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right)-\frac{k}{x-x_*}-\frac{1}{C(x-x_*)^k}+\frac{1}{2}\frac{1}{C(x-x_*)^k}+\frac{1}{2}\left(\frac{1}{x_*}+\frac{1}{x_*-1}\right),$$

which implies that

$$Ck(k-1) = \frac{Ck^2}{2};$$

that is, $k = 2$ for an arbitrary $C \neq 0$.

Thus, we have found that the function $y(x) - x$ can have only first-order zeros if $\delta \neq 1/2$ and only second-order zeros if $\delta = 1/2$.

In a similar way, we can show that $y(x)$ can have only first-order poles if $\alpha \neq 0$ and second-order poles for $\alpha = 0$. This function can have only first-order zeros if $\beta \neq 0$ and second-order zeros if $\beta = 0$. The function $y(x) - 1$ can have only first-order zeros if $\gamma \neq 0$ and second-order zeros if $\gamma = 0$.

Now, we derive auxiliary equations for all the types of critical points of the Painlevé VI equation using the successive elimination of singularities. Only the main stages of this derivation are presented, and the rather cumbersome calculations are not shown.

2. SYSTEM OF EQUATIONS IN A NEIGHBORHOOD OF A POLE

Let x_* be a pole of a solution $y(x)$. We examine separately two cases: $\alpha \neq 0$, then x_* is a first-order pole; $\alpha = 0$, then x_* is a second-order pole.

1. The Basic Case: $\alpha \neq 0$.

In a deleted neighborhood of x_*, we define the functions

$$u(x) = \frac{1}{y(x)}, \quad v(x) = \frac{y'(x)}{y^2(x)}. \tag{2.1}$$

Then

$$y(x) = \frac{1}{u(x)}, \quad y' = \frac{v(x)}{u^2(x)}. \tag{2.2}$$

The functions $u(x)$ and $v(x)$, being analytically continued to the entire neighborhood of x_*, satisfy in this neighborhood the system of equations

$$u' = -v, \tag{2.3}$$

$$uv' = \frac{v^2}{2}\left(-3+\frac{1}{1-u}+\frac{1}{1-xu}\right)-uv\left(\frac{1}{x}+\frac{1}{x-1}+\frac{u}{1-xu}\right)$$
$$+\frac{(1-u)(1-xu)}{x^2(x-1)^2}\left(\alpha+\beta xu^2+\gamma\frac{x-1}{(1-u)^2}u^2+\frac{\delta x(x-1)}{(1-xu)^2}u^2\right). \tag{2.4}$$

At x_*, we have the equality $u(x_*) = 0$, and Eq. (2.4) has a singularity at this point. It follows from this equation that $v^2(x_*) = 2\alpha/x_*^2(x_* - 1)^2$. Then Eq. (2.3) implies that $u'(x_*) \neq 0$. Hence, in a certain neighborhood of x_*, there exists an analytic function $w(x)$ such that

$$v(x) = v_*(x) + u(x)w(x), \tag{2.5}$$

where $v_*(x) = \sqrt{2\alpha}/x(x-1)$ (we choose a definite root later when transiting to the auxiliary equations).

Using representation (2.5) and Eq. (2.4), we obtain a differential equation for the function $w(x)$; namely,

$$\begin{aligned} w' &= \frac{w^2}{2}\left(-1+\frac{1}{1-u}+\frac{1}{1-xu}\right)-w\left(\frac{1}{x}+\frac{1}{x-1}+\frac{u}{1-xu}\right) \\ &+\frac{\sqrt{2\alpha}}{x(x-1)}\left(\frac{w}{1-u}-\frac{1-xw}{1-xu}\right)+\frac{\alpha}{x^2(x-1)^2}\left(\frac{1}{1-u}+x+\frac{x^2}{1-xu}\right) \\ &+\frac{(1-u)(1-xu)}{x^2(x-1)^2}\left(\beta x+\frac{\gamma(x-1)}{(1-u)^2}+\frac{\delta x(x-1)}{(1-xu)^2}\right). \end{aligned} \tag{2.6}$$

This equation has no singularity at x_*.

Thus, using the change of variables (2.1) and (2.5), we reduced Eq. (1.1) to the equivalent system (2.3), (2.6), which has a singularity neither at x_* nor in a neighborhood of this point.

Relations (2.1) and (2.2) are used for obtaining initial conditions in the transition from Eq. (1.1) to the auxiliary system and in the reverse transition.

Note that, for the transition to the auxiliary system of equations, we choose the value of $v_*(x)$ that is closer to the value of the function $v(x) = y'(x)/y^2(x)$ at the corresponding point.

2. The Particular Case: $\alpha = 0$.

In this case, we define the functions

$$u(x) = \frac{2y(x)}{y'(x)}, \quad v(x) = \frac{y'^2(x)}{4y^3(x)} \tag{2.7}$$

in a deleted neighborhood of x_* and then analytically continue them to the entire neighborhood. It follows that

$$y(x) = \frac{1}{u^2 v}, \quad y'(x) = \frac{2}{u^3 v}. \tag{2.8}$$

For the functions $u(x)$ and $v(x)$, we obtain the following auxiliary system of equations:

$$\begin{aligned} u' &= 1-\frac{1}{1-u^2v}-\frac{1}{1-xu^2v}+u\left(\frac{1}{x}+\frac{1}{x-1}+\frac{u^2v}{1-xu^2v}\right) \\ &-\frac{u^2(1-u^2v)(1-xu^2v)}{2x^2(x-1)^2}\left(\beta x+\gamma\frac{x-1}{(1-u^2v)^2}+\frac{\delta x(x-1)}{(1-xu^2v)^2}\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} v' &= 2uv^2\left(\frac{1}{1-u^2v}+\frac{x}{1-xu^2v}\right)-2v\left(\frac{1}{x}+\frac{1}{x-1}+\frac{u^2v}{1-xu^2v}\right) \\ &+\frac{uv(1-u^2v)(1-xu^2v)}{x^2(x-1)^2}\left(\beta x+\gamma\frac{x-1}{(1-u^2v)^2}+\frac{\delta x(x-1)}{(1-xu^2v)^2}\right). \end{aligned} \tag{2.10}$$

It is equivalent to Eq. (1.1) and has no singularity at x_*.

Formulas (2.7) and (2.8) are used in the transition to auxiliary equations (2.9) and (2.10) and in the reverse transition.

3. SYSTEM OF EQUATIONS IN A NEIGHBORHOOD OF A ZERO OF $y(x)$

Let x_* be a zero of $y(x)$. Here, as was already indicated, we must distinguish two cases: $\beta \neq 0$, then x_* is a first-order zero; $\beta = 0$, then x_* is a second-order zero.

1. The Basic Case: $\beta \neq 0$.

In a certain neighborhood of x_*, we define the functions

$$u(x) = y(x), \quad v(x) = y'(x). \tag{3.1}$$

Then these functions satisfy the system of equations

$$u' = v, \tag{3.2}$$

$$uv' = \frac{v^2}{2}\left(1 + \frac{u}{u-1} + \frac{u}{u-x}\right) - uv\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)$$
$$+ \frac{(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha u^2 + \beta x + \gamma\frac{x-1}{(u-1)^2}u^2 + \frac{\delta x(x-1)}{(u-x)^2}u^2\right). \tag{3.3}$$

Since $u(x_*) = 0$, Eq. (3.3) implies that $v^2(x_*) = -2\beta/(x_* - 1)^2$. Consequently, $u'(x_*) = v(x_*) \neq 0$, and, in a certain neighborhood of x_*, there exists an analytic function $w(x)$ such that

$$v(x) = v_*(x) + u(x)w(x), \tag{3.4}$$

where $v_*(x) = \sqrt{-2\beta}/(x-1)$ (a definite root is chosen when transiting to the auxiliary equations). Using this representation and formulas (3.2) and (3.3), we obtain the following equation for the function $w(x)$:

$$w' = \frac{w^2}{2}\left(-1 + \frac{u}{u-1} + \frac{u}{u-x}\right) - w\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)$$
$$+ \frac{\sqrt{-2\beta}}{(x-1)(u-x)}\left(w\left(\frac{u-x}{u-1} + 1\right) - \frac{1}{x}\right) + \frac{\beta}{(x-1)^2}\left(\frac{1}{x} - \frac{1}{u-1} - \frac{1}{x(u-x)}\right) \tag{3.5}$$
$$+ \frac{(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \frac{\gamma(x-1)}{(u-1)^2} + \frac{\delta x(x-1)}{(u-x)^2}\right).$$

This equation has no singularity at x_*.

Thus, using the change of variables (3.1) and (3.4), we reduced the original equation (1.1) to system (3.2), (3.5), which has a singularity neither at x_* nor in a neighborhood of this point.

For the transition to this auxiliary system of equations, we choose the value of $v_*(x)$ that is closer to the value of the function $v(x) = y'(x)$.

Relations (3.1) are used for obtaining initial conditions in the transition from the original equation to the auxiliary system and in the reverse transition.

2. The Particular Case: $\beta = 0$.

To obtain an auxiliary system, we define the functions

$$u(x) = \frac{2y(x)}{y'(x)}, \quad v(x) = \frac{y'^2(x)}{4y(x)}, \tag{3.6}$$

which implies the expressions

$$y(x) = u^2(x)v(x), \quad y'(x) = 2u(x)v(x). \tag{3.7}$$

Then, in a certain neighborhood of x_*, Eq. (1.1) is equivalent to the following auxiliary system of equations

$$u' = 1 - u^2v\left(\frac{1}{u^2v-1} + \frac{1}{u^2v-x}\right) + u\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u^2v-x}\right)$$
$$- u^2\frac{(u^2v-1)(u^2v-x)}{2x^2(x-1)^2}\left(\alpha + \frac{\gamma(x-1)}{(u^2v-1)^2} + \frac{\delta x(x-1)}{(u^2v-x)^2}\right),$$
$$v' = 2uv^2\left(\frac{1}{u^2v-1} + \frac{1}{u^2v-x}\right) - 2v\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u^2v-x}\right) \tag{3.8}$$
$$+ \frac{uv(u^2v-1)(u^2v-x)}{x^2(x-1)^2}\left(\alpha + \frac{\gamma(x-1)}{(u^2v-1)^2} + \frac{\delta x(x-1)}{(u^2v-x)^2}\right),$$

having no singularities.

Relations (3.6) and (3.7) are used in the transition from Eq. (1.1) to system (3.8) and in the reverse transition.

4. SYSTEM OF EQUATIONS IN A NEIGHBORHOOD OF A ZERO OF THE FUNCTION $y(x) - 1$

Let x_* be a zero of the function $y(x) - 1$; that is, $y(x_*) = 1$. Then, as shown above, x_* is a first-order zero if $\gamma \neq 0$ and a second-order zero if $\gamma = 0$.

1. The Basic Case: $\gamma \neq 0$.

In this case, we define the functions

$$u(x) = y(x) - 1, \quad v(x) = y'(x). \tag{4.1}$$

These functions satisfy the system of equations

$$u' = v, \tag{4.2}$$

$$\begin{aligned} uv' &= \frac{v^2}{2}\left(1 + \frac{u}{u+1} + \frac{u}{u-x+1}\right) - uv\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x+1}\right) \\ &+ \frac{(u+1)(u-x+1)}{x^2(x-1)^2}\left(\alpha u^2 + \frac{\beta x u^2}{(u+1)^2} + \gamma(x-1) + \frac{\delta x(x-1)u^2}{(u-x+1)^2}\right). \end{aligned} \tag{4.3}$$

Equation (4.3) has a singularity at the point x_*. Since $u(x_*) = 0$, this equation implies the relation

$$v^2(x_*) = 2\gamma/x_*^2.$$

Therefore, $u'(x_*) \neq 0$ and, in a certain neighborhood of x_*, it holds that

$$v(x) = v_*(x) + uw, \tag{4.4}$$

where $v_*(x) = \sqrt{2\gamma}/x$ and $w(x)$ is an analytic function. A definite root is chosen when transiting to the auxiliary equations.

Using (4.3) and (4.4), we obtain an equation for the function $w(x)$, namely,

$$\begin{aligned} w' &= \frac{w^2}{2}\left(-1 + \frac{u}{u+1} + \frac{u}{u-x+1}\right) - w\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x+1}\right) \\ &+ \frac{\sqrt{2\gamma}}{x(u-x+1)}\left(w\left(2 - \frac{x}{u+1}\right) - \frac{1}{x-1}\right) + \frac{\gamma}{x^2}\left(\frac{1}{x-1} - \frac{1}{u+1} + \frac{1}{(x-1)(u-x+1)}\right) \\ &+ \frac{(u+1)(u-x+1)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u+1)^2} + \frac{\delta x(x-1)}{(u-x+1)^2}\right). \end{aligned} \tag{4.5}$$

This equation has no singularity at x_*.

Thus, in a neighborhood of a first-order zero of the function $y(x) - 1$, the original equation is equivalent to the auxiliary system of equations (4.2), (4.5) having no singularities.

Relations (4.1) are used in the transitions between the original equation and the auxiliary system.

2. The Particular Case: $\gamma = 0$.

To obtain auxiliary equations in this case, we define the functions

$$u(x) = \frac{2(y(x) - 1)}{y'(x)}, \quad v(x) = \frac{y'^2(x)}{4(y(x) - 1)}, \tag{4.6}$$

which implies the relations

$$y(x) = 1 + u^2(x)v(x), \quad y'(x) = 2u(x)v(x). \tag{4.7}$$

In the usual fashion, we derive equations for the functions introduced in (4.6), namely,

$$u' = -\frac{u^2 v}{u^2 v + 1 - x} + \frac{1}{u^2 v + 1} + u\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u^2 v + 1 - x}\right)$$
$$-\frac{u^2(u^2 v + 1)(u^2 v + 1 - x)}{2x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u^2 v + 1)^2} + \frac{\delta x(x-1)}{(u^2 v + 1 - x)^2}\right), \tag{4.8}$$

$$v' = 2uv^2\left(\frac{1}{u^2 v + 1} + \frac{1}{u^2 v + 1 - x}\right) - 2v\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u^2 v + 1 - x}\right)$$
$$+\frac{uv(u^2 v + 1)(u^2 v + 1 - x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u^2 v + 1)^2} + \frac{\delta x(x-1)}{(u^2 v + 1 - x)^2}\right). \tag{4.9}$$

Equations (4.8) and (4.9) have no singularities in a neighborhood of x_*.

Relations (4.6) and (4.7) are used in the transition to the auxiliary system and in the reverse transition.

Note an important property of Eq. (1.1). Suppose that, in this equation, we perform the change of variables according to the formulas $x = 1 - \hat{x}$, $y(x) = 1 - \hat{y}(\hat{x})$, $\hat{\alpha} = \alpha$, $\hat{\beta} = -\gamma$, $\hat{\gamma} = -\beta$, and $\hat{\delta} = \delta$. Then the Painlevé VI equation for the function $\hat{y}(\hat{x})$ is obtained. Furthermore, the zeros of $\hat{y}(\hat{x})$ correspond to the zeros of $y(x) - 1$, and vise versa. Consequently, the indicated change of the variables and parameters in the equation allows us to derive the auxiliary equations found in this subsection from the corresponding equations for a zero of $y(x)$. This property can be used for an additional check in the derivation of auxiliary equations.

5. SYSTEM OF EQUATIONS IN A NEIGHBORHOOD OF A ZERO OF THE FUNCTION $y(x) - x$

Let x_* be a zero of the function $y(x) - x$; that is, $y(x_*) = x_*$. Then, as shown above, x_* is a first-order zero if $\delta \neq 1/2$ and a second-order zero if $\delta = 1/2$.

1. The Basic Case: $\delta \neq 1/2$.

In this case, we define the functions

$$u(x) = y(x) - x, \quad v(x) = y'(x) - 1. \tag{5.1}$$

These functions satisfy the system of equations

$$u' = v, \tag{5.2}$$

$$uv' = \frac{(v+1)^2}{2}\left(1 + \frac{u}{u+x} + \frac{u}{u+x-1}\right) - (v+1)\left(1 + \frac{u}{x} + \frac{u}{x-1}\right)$$
$$+\frac{(u+x)(u+x-1)}{x^2(x-1)^2}\left(\alpha u^2 + \frac{\beta x u^2}{(u+x)^2} + \frac{\gamma(x-1)u^2}{(u+x-1)^2} + \delta x(x-1)\right). \tag{5.3}$$

Equation (5.3) has a singularity at the point x_*. Since $u(x_*) = 0$, this equation implies the relation

$$v^2(x_*) = 1 - 2\delta.$$

Therefore, $u'(x_*) \neq 0$ and, in a certain neighborhood of x_*, it holds that

$$v(x) = v_*(x) + uw, \tag{5.4}$$

where $v_*(x) = \sqrt{1 - 2\delta}$ and $w(x)$ is an analytic function. As before, a definite root is chosen when transiting to the auxiliary equations.

Using (5.3) and (5.4), we obtain an equation for the function $w(x)$, namely,

$$w' = \frac{w^2}{2}\left(-1 + \frac{u}{u+x} + \frac{u}{u+x-1}\right) + \sqrt{1-2\delta}\left(\frac{1}{u+x} + \frac{1}{u+x-1}\right)w$$
$$-\left(\frac{1}{x(u+x)} + \frac{1}{(x-1)(u+x-1)}\right)(v+1-\delta) + \frac{\delta}{x(x-1)} \tag{5.5}$$
$$+\frac{(u+x)(u+x-1)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u+x)^2} + \frac{\gamma(x-1)}{(u+x-1)^2}\right).$$

This equation has no singularity at x_*.

Thus, in a neighborhood of a first-order zero of the function $y(x) - x$, the original equation is equivalent to the auxiliary system of equations (5.2), (5.5) having no singularities.

As usual, relations (5.1) and (5.4) are used in the transitions between the original equation and the auxiliary system.

2. The Particular Case: $\delta = 1/2$.

To obtain auxiliary equations in this case, we define the functions

$$u(x) = 2\frac{y(x)-x}{y'(x)-1}, \quad v(x) = \frac{1}{4}\frac{(y'(x)-1)^2}{y(x)-x}, \tag{5.6}$$

which implies the relations

$$y(x) = u^2(x)v(x) + x, \quad y'(x) = 2u(x)v(x) + 1. \tag{5.7}$$

The functions introduced in (5.6) satisfy the equations

$$u' = 1 - \frac{1}{2v}\left\{\frac{1}{x-1} + \frac{u^2v-1}{2x(x-1)} + \frac{(1+2uv)^2}{2}\left(\frac{1}{u^2v+x} + \frac{1}{u^2v+x-1}\right) - (1+2uv)\left(\frac{1}{x} + \frac{1}{x-1}\right)\right\}$$
$$-\frac{u^2(u^2v+x)(u^2v+x-1)}{2x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u^2v+x)^2} + \frac{\gamma(x-1)}{(u^2v+x-1)^2}\right), \tag{5.8}$$

$$v' = \frac{uv}{2}\left(\frac{1}{x(x-1)} - \frac{1}{x(u^2v+x)} - \frac{1}{(x-1)(u^2v+x-1)}\right)$$
$$+2v(uv+1)\left(\frac{1}{u^2v+x} + \frac{1}{u^2v+x-1}\right) - 2v\left(\frac{1}{x} + \frac{1}{x-1}\right) \tag{5.9}$$
$$+\frac{uv(u^2v+x)(u^2v+x-1)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{(u^2v+x)^2} + \frac{\gamma(x-1)}{(u^2v+x-1)^2}\right).$$

Equations (5.8) and (5.9) have no singularities in a neighborhood of x_*.

Relations (5.6) and (5.7) are used in the transition to the auxiliary system and in the reverse transition.

Thus, we have derived auxiliary equations for each of the eight types of critical points of the Painlevé VI equation.

The following remarks apply to all the derived equations.

1. In all cases, the critical point satisfies the condition $u(x_*) = 0$. Since, in each case, we have $u'(x_*) \neq 0$, the position of this point can be determined in a numerically stable way.

2. Equation (1.1) has a one-parameter family of solutions at each second-order critical point and two one-parameter families of solutions at each first-order critical point. For each first-order point, the choice of a family corresponds to the choice of a value for the square root.

3. An interesting fact is that, by solving the Cauchy problem for any auxiliary system subject to the initial conditions

$$u(x_0) = 0, \quad w(x_0) = w_0,$$

we select a particular solution to Eq. (1.1) from the above families of solutions that pass through the corresponding critical point. Thus, such a Cauchy problem for an auxiliary system can replace the Cauchy problem for the original equation (1.1) and makes it possible to solve the latter equation starting from the critical point.

4. Equations (2.10), (3.8), (4.9), and (5.9), corresponding to the cases $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $\delta = 1/2$, respectively, have no singularities. Note that, in these cases, the initial conditions for the corresponding auxiliary systems, selecting specific solutions to Eq. (1.1), are

$$u(x_0) = 0, \quad v(x_0) = v_0 \neq 0.$$

6. CRITERIA FOR THE TRANSITION TO AUXILIARY SYSTEMS OF EQUATIONS AND FOR THE REVERSE TRANSITION

In practice, the numerical solution of the problem under discussion can be organized as follows. We use the original equation (1.1) as long as we are not too close to a critical point. Then, in accordance with the type of this point, we transit to the corresponding auxiliary system. Having gone across the critical point, we return to the original equation and continue the calculations. To obtain the initial conditions required for such transitions, we use the formulas relating the original and auxiliary variables.

In order to implement this method, it is very important to know the best moment for transiting to the auxiliary systems and the one for the reverse transition. One should consider that calculations with the original equation are undesirable in a vicinity of a critical point, while calculations with the auxiliary equations are undesirable when one is far from critical points. Both the former and the latter can result in a great loss of accuracy.

The transition criteria formulated below are based on the following conventions. A transition from the original equation to an auxiliary system is performed if the values $|u(x)|$ and $|v(x) - v_*(x)|$ become small. The reverse transition to the original equation is done when at least one of the above conditions is violated. Whether or not the indicated values are small is specified by some experimentally found numbers ε_i ($0 < \varepsilon_i < 1$). The experience gathered in calculations with the Painlevé I–V equations, as well as the results obtained for the Painlevé VI equation, which are presented below, suggest that the same numbers ε_i can be used for all the "typical" variants.

Let us state the specific formulations of the transition criteria for each case under discussion.

First-order pole.

Here, $v_*(x) = \sqrt{2\alpha}/x(x-1)$.

The transition from y and y' to u and w is made if

$$\varepsilon_1|y| > 1 \quad \text{and} \quad \left|y' - v_*y^2\right| < \varepsilon_2\left|v_*y^2\right|. \tag{6.1}$$

The reverse transition from u and w to y and y' is performed if

$$|u| > \varepsilon_1 \quad \text{or} \quad |v - v_*| > \varepsilon_2|v_*|. \tag{6.2}$$

As was already noted, a well-defined value of $v_*(x)$ is chosen for the transition to the corresponding auxiliary system. Namely, this is the value that is closer to the value of the function $v(x) = y'(x)/y^2(x)$.

Second-order pole.

The transition from y and y' to u and v is made if

$$|2y| < \varepsilon_3|y'| \quad \text{and} \quad \varepsilon_4|y| > 1. \tag{6.3}$$

The reverse transition from u and v to y and y' is performed if

$$|u| > \varepsilon_3 \quad \text{or} \quad \left|u^2 v\right| > \varepsilon_4. \tag{6.4}$$

The value of $v(x_*)$ in this case can be arbitrary. Consequently, to verify that a pole occurs, we add the condition that $|y(x)|$ be sufficiently large.

First-order zero of $y(x)$ ($\beta \neq 0$).

Here, $v_*(x) = \dfrac{\sqrt{-2\beta}}{x-1}$.

The transition from y and y' to u and w is made if

$$|y| < \varepsilon_5 \quad \text{and} \quad |y' - v_*| < \varepsilon_6 |v_*|. \tag{6.5}$$

The reverse transition from u and w to y and y' is performed if

$$|u| > \varepsilon_5 \quad \text{or} \quad |v - v_*| > \varepsilon_6 |v_*|. \tag{6.6}$$

Again, a definite value of $v_*(x)$ is chosen for the transition to the corresponding auxiliary system. Namely, this is the value that is closer to the value of the function $v(x) = y'(x)$.

Second-order zero of $y(x)$ ($\beta = 0$).

Here, the value of $v(x_*)$ is also not defined; therefore, the smallness condition for the function u is supplemented by the condition that y be small.

The transition from y and y' to u and v is made if

$$2|y| < \varepsilon_7 |y'| \quad \text{and} \quad |y| < \varepsilon_8. \tag{6.7}$$

The reverse transition from u and v to y and y' is performed if

$$|u| > \varepsilon_7 \quad \text{or} \quad |u^2 v| > \varepsilon_8. \tag{6.8}$$

First-order zero of $(y(x) - 1)$ ($\gamma \neq 0$).

Here, $v_* = \sqrt{2\gamma}/x$.

The transition from y and y' to u and w is made if

$$|y - 1| < \varepsilon_9 \quad \text{and} \quad |y' - v_*| < \varepsilon_{10} |v_*|. \tag{6.9}$$

The reverse transition from u and w to y and y' is performed if

$$|u| > \varepsilon_9 \quad \text{or} \quad |v - v_*| > \varepsilon_{10} |v_*|. \tag{6.10}$$

When transiting to the corresponding auxiliary system, one should choose the value of v_* that is closer to the value of the function $v(x) = y'(x)$.

Second-order zero of $(y(x_*) - 1)$ ($\gamma = 0$).

Since the value of $v(x_*)$ is not defined, the smallness condition for the function u is supplemented by the condition that y be close to unity.

The transition from y and y' to u and v is made if

$$2|y - 1| < \varepsilon_{11} |y'| \quad \text{and} \quad |y - 1| < \varepsilon_{12}. \tag{6.11}$$

The reverse transition from u and v to y and y' is performed if

$$|u| > \varepsilon_{11} \quad \text{or} \quad |u^2 v| > \varepsilon_{12}. \tag{6.12}$$

First-order zero of $(y(x) - x)$ ($\delta \neq 1/2$).

Here, $v_* = \sqrt{1 - 2\delta}$.

The transition from y and y' to u and w is made if

$$|y - x| < \varepsilon_{13} \quad \text{and} \quad |y' - 1 - v_*| < \varepsilon_{14} |v_*|. \tag{6.13}$$

The reverse transition from u and w to y and y' is performed if

$$|u| > \varepsilon_{13} \quad \text{or} \quad |v - v_*| > \varepsilon_{14} |v_*|. \tag{6.14}$$

When transiting to the corresponding auxiliary system, one should choose the value of v_* that is closer to the value of the function $v(x) = y'(x) - 1$.

Second-order zero of $(y(x) - x)$ ($\delta = 1/2$).

Since the value of $v(x_*)$ is not defined, the smallness condition for the function u is supplemented by the condition that y be close to the value of x.

The transition from y and y' to u and v is made if

$$2|y - x| < \varepsilon_{15} |y' - 1| \quad \text{and} \quad |y - x| < \varepsilon_{16}. \tag{6.15}$$

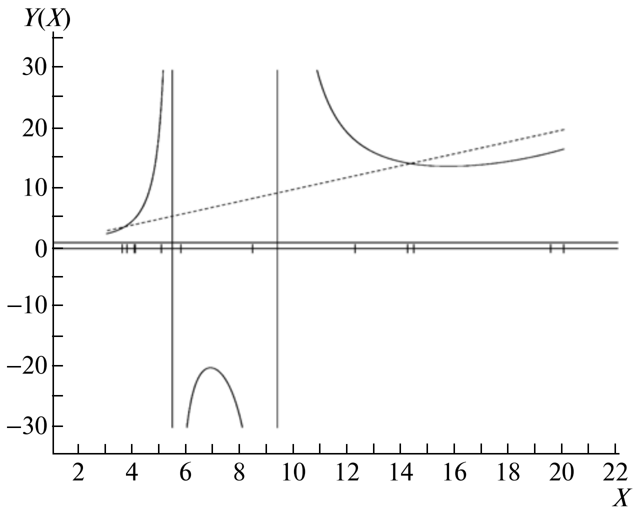

Fig. 1. $\alpha = 2$, $\beta = -2.5$, $\gamma = 5$, $\delta = -1$, $y(x_0) = 2.5$, $y'(x_0) = 1$.

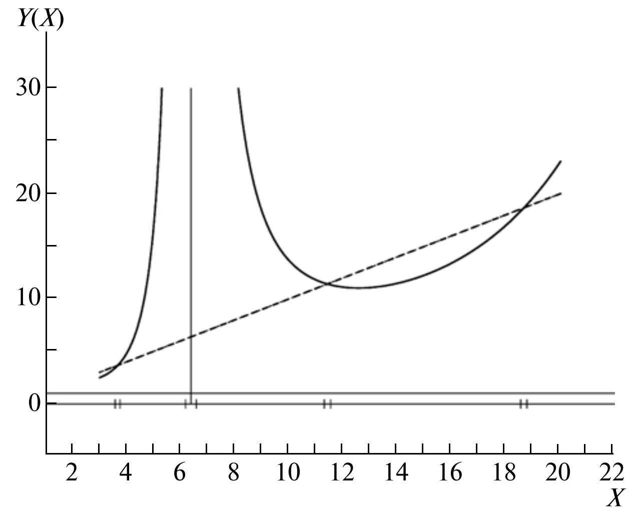

Fig. 2. $\alpha = 0$, $\beta = -2.5$, $\gamma = 5$, $\delta = -1$, $y(x_0) = 2.5$, $y'(x_0) = 1$.

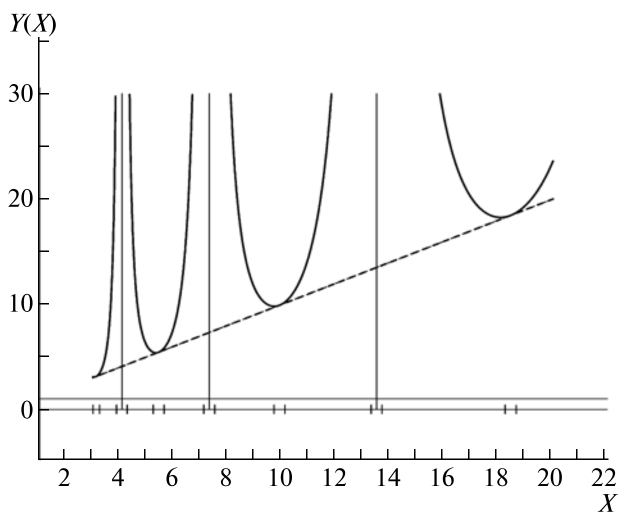

Fig. 3. $\alpha = 0$, $\beta = -1.5$, $\gamma = 3$, $\delta = 0.5$, $y(x_0) = 3.1$, $y'(x_0) = -1$.

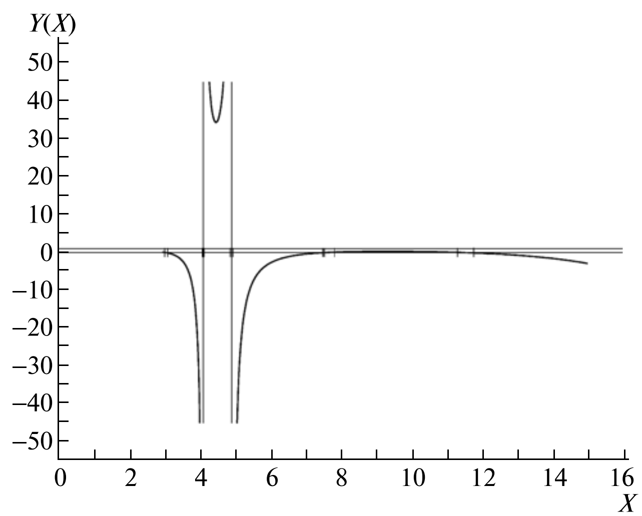

Fig. 4. $\alpha = 3$, $\beta = -5$, $\gamma = 2$, $\delta = -0.5$, $y(x_0) = 0.1$, $y'(x_0) = -1$.

The reverse transition from u and v to y and y' is performed if

$$|u| > \varepsilon_{15} \quad \text{or} \quad |u^2 v| > \varepsilon_{16}. \tag{6.16}$$

In all the formulas (6.1) to (6.16), the constants ε_i ($i = 1, \ldots, 16$) are positive numbers chosen experimentally. In our practice, these constants were the same for all calculations. This makes the above formulations a universal criterion.

The following remark is in order here. The method under discussion employs fairly crude estimates of the positions of critical points. In practice, this may lead to that the transition criterion for some critical point is incorrectly used. However, the results of calculations will not be affected by such an event because the original equation and the corresponding auxiliary system are equivalent in a certain neighborhood of this critical point.

7. NUMERICAL RESULTS

Here, we discuss certain examples of solving the Painlevé VI equation for various initial data and values of the parameters in this equation.

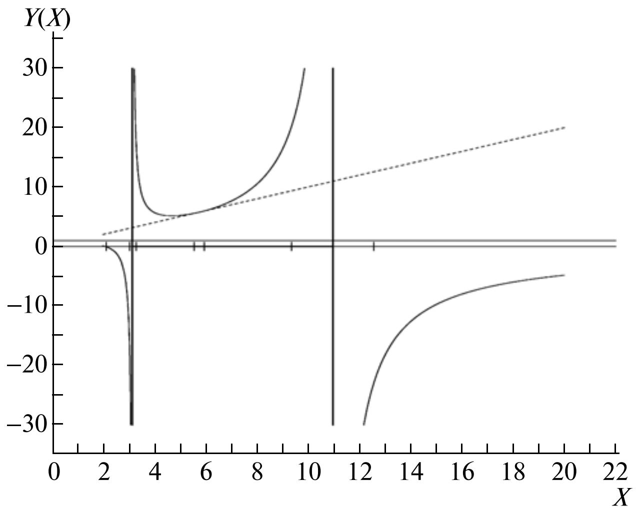

Fig. 5. $\alpha = 5$, $\beta = -1$, $\gamma = 0$, $\delta = 0.5$, $y(x_0) = 0.1$, $y'(x_0) = -1$.

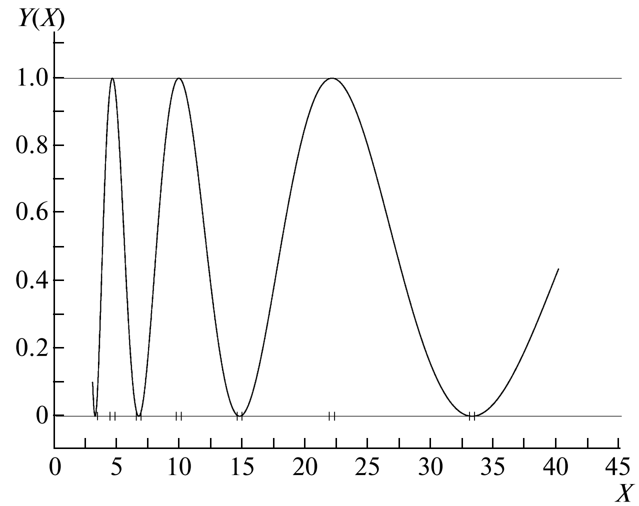

Fig. 6. $\alpha = 5$, $\beta = 0$, $\gamma = 0$, $\delta = 0.5$, $y(x_0) = 0.1$, $y'(x_0) = -1$.

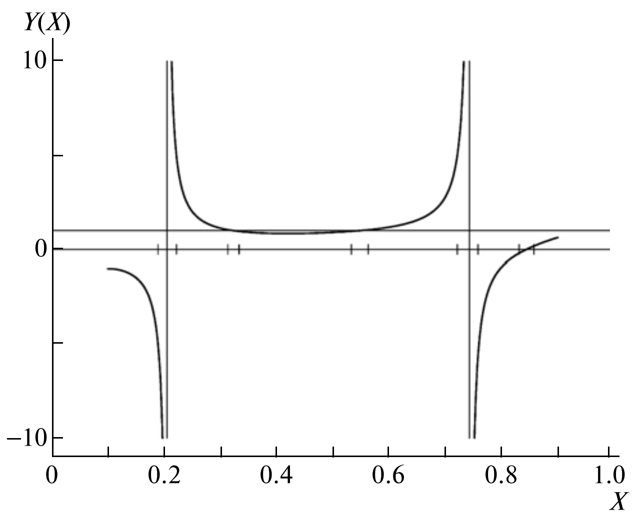

Fig. 7. $\alpha = 2$, $\beta = -2.5$, $\gamma = 1$, $\delta = -1$, $y(x_0) = -1$, $y'(x_0) = 1$.

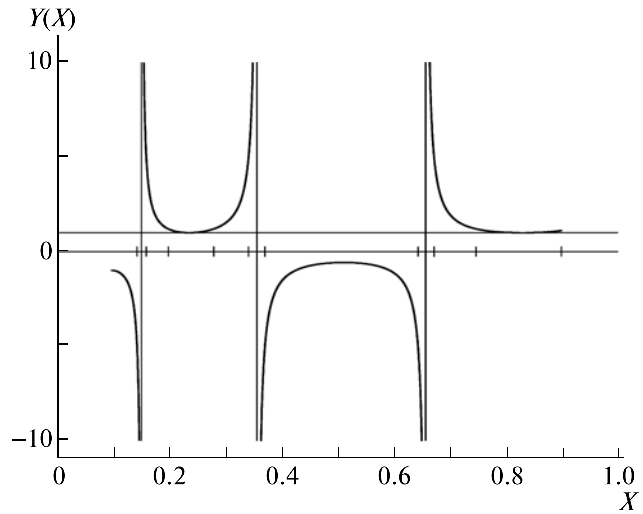

Fig. 8. $\alpha = 5$, $\beta = 0$, $\gamma = 0$, $\delta = -0.5$, $y(x_0) = -1$, $y'(x_0) = 1$.

For integrating differential equations, we used the conventional fourth-order Runge–Kutta method with a constant step size. As usual, the accuracy of calculations was estimated by comparing the results corresponding to reduced step sizes.

For all the problems treated, we took the unified value $\varepsilon_i = 0.2$ ($i = 1, \ldots, 16$) for all the constants. We also conducted calculations with other values of these constants. The results of such tests agree closely with each other, which indicates the numerical stability of the proposed method.

For greater simplicity, almost all the problems were solved on the same interval [2, 20]. For the results discussed below, the number N of grid points on a given interval was 32000.

In order to demonstrate the capabilities of our method, we present the graphs of curves containing critical points of all possible types (see Figs. 1–8). The marks on the axis x correspond to the intervals for which the calculations are governed by the auxiliary systems. For greater clearness, the lines $y = 1$ and $y = x$ are also drawn on the figures.

We make several remarks on these figures.

Table

N	Pole	Pole	$y = 0$	$y = x$
2000	3.155769	10.97722	2.087235	5.761728
4000	3.155701	10.97739	2.087254	5.761654
8000	3.155699	10.97737	2.087260	5.761649
16000	3.155698	10.97737	2.087260	5.761649
32000	3.155697	10.97737	2.087261	5.761649

Figure 1 corresponds to a nonzero set of the parameters ($\alpha = 2$, $\beta = -2.5$, $\gamma = 5$, and $\delta = -1$). The initial conditions are $y_0 = 2.5$ and $y_0' = 1$. There are two first-order poles and two first-order zeros of the function $y - x$ on this figure.

Figure 2 corresponds to the case where $\alpha = 0$, whereas the other data are the same as in Fig. 1. The figure contains a second-order pole and three first-order zeros of the function $y - x$. Here, one can see how two first-order poles convert into a single second-order pole.

Figure 3 corresponds to the parameters $\alpha = 0$, $\beta = 1.5$, $\gamma = 3$, and $\delta = 0.5$. The initial conditions are $y_0 = 3.1$ and $y_0' = -1$. There are three second-order poles and four second-order zeros of the function $y - x$ on this figure.

Figure 4 corresponds to the parameters $\alpha = 3$, $\beta = -5$, $\gamma = 2$, and $\delta = -0.5$. The initial conditions are $y_0 = 0.1$ and $y_0' = -1$. The figure contains two first-order poles and three first-order zeros of y.

Figure 5 corresponds to the parameters $\alpha = 5$, $\beta = -1$, $\gamma = 0$, and $\delta = 0.5$. The initial conditions are $y_0 = 0.1$ and $y_0' = -1$. There are two first-order poles, a first-order zero of y, and a second-order zero of the function $y - x$ on this figure.

Figure 6 corresponds to the case where $\beta = 0$, whereas the other data are the same as in Fig. 5. The figure contains four second-order zeros of y and three second-order zeros of the function $y - 1$. There are no poles in this example. The curve is squeezed in the strip $0 \leqslant y \leqslant 1$.

Figures 7 and 8 correspond to the examples in which the equation was solved on the interval [0.1, 0.9] whose endpoints are close to the deleted points $x = 0$ and $x = 1$.

Figure 7 corresponds to the parameters $\alpha = 2$, $\beta = -2.5$, $\gamma = 1$, and $\delta = -1$. The initial conditions are $y_0 = -1$ and $y_0' = 1$. There are two first-order poles, a first-order zero of y, and two first-order zeros of the function $y - 1$ on this figure.

Figure 8 corresponds to the parameters $\alpha = 5$, $\beta = 0$, $\gamma = 0$, and $\delta = -0.5$. The initial conditions are $y_0 = -1$ and $y_0' = 1$. The figure contains three first-order poles and two second-order zeros of the function $y - 1$.

The above figures demonstrate that changes in the numerical data can radically change the solution. This clearly shows the mobility of critical points.

We emphasize that the presented results were numerically verified by repeating the calculations with a smaller step size.

Let us give an example of numerical data. Consider, for instance, Fig. 5, which contains four critical points of three types, namely, two first-order poles, a first-order zero of y, and a second-order zero of the function $y - x$.

The numerical values of the positions of these points for grids of different density are presented in Table 1. Note that these values are obtained by using linear interpolation between the grid points at which the function $u(x)$ changes its sign. Since $u'(x)$ is close to a nonzero constant in a neighborhood of a critical point of any type, $u(x)$ is almost linear in such a neighborhood. Thus, linear interpolation is able to determine the positions of these points to high accuracy.

The numerical tests performed by the authors confirm that the proposed method is efficient and convenient in practical applications.

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